Quasiparticle pair creation in unstable superflow

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Landau's instability mechanism in superflow is considered with special attention given to the role of nonuniformity in the flow. Linear stability analysis applied to the first in a series of approximate microscopic equations for the superfluid reveals a growth rate for Landau's instability proportional to the shear in the flow. In a quasiparticle description, the shear acts as a source of particle pair creation. The observation of roton-pair creation in experiments with electron bubbles in helium is offered as evidence of this phenomenon.

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I. INTRODUCTION

Landau's classical argument [1], which relates the onset velocity for dissipation in a superfluid to the dispersion of elementary excitations, relies on a Galilean transformation. An example of a system to which such a transformation can be applied is a superfluid in an infinite cylindrical container. In its ground state, the superfluid is at rest with respect to the cylinder; excitations, characterized by a momentum $\mathbf{k}$ along the axis of the cylinder, have positive energy $\epsilon_{\mathbf{k}}$. In any other frame of uniform motion along the cylinder axis, the energy of excitations is given by $\epsilon_{\mathbf{k}}+\mathbf{v}\cdot\mathbf{k}$, where $\mathbf{v}$ is the apparent flow velocity seen by the observer. Above a certain critical flow velocity $\mathbf{v}_L$ (or state of motion of the observer), the energies of some excitations become negative; apparently the stability of the system is jeopardized since its energy can be lowered by spontaneously creating excitations.

The same Galilean transformations can, however, also transform a state of flow into a state of rest. Since, by definition, the ground state *is* stable, this implies that in fact all states of uniform flow are stable as well. After all, the state of motion of the observer should have no influence on the stability of what is being observed. Alternatively, we can be quite sure that in order for a flow to be unstable it must *not* be possible to transform it to the ground state. That this is generally true in practice is clear when we consider the walls of the cylinder. Unless the walls are perfectly smooth, any flow will have some nonuniform component that no Galilean transformation can eliminate. Although Landau's argument correctly identifies the excitations or modes responsible for the instability, the growth of such modes requires nonuniformity or shear in the superfluid.

This paper reexamines Landau's instability problem with a tool that is able to deal with general states of flow. The three main results are as follows: (1) a Schrödinger-like equation for modes in nonuniform flow, (2) a formula for the growth of such modes, and (3) the interpretation of this instability as a quasiparticle pair-creation process. The tool, called *continuous collapse dynamics* [2], is an approximation that transforms the time-dependent Schrödinger equation for many degrees of freedom into a nonlinear evolution equation involving relatively few degrees of freedom. Standard linear stability analysis applied to the nonlinear equation leads to the above stated results. A quasiparticle interpretation of these results is offered as an explanation of the puzzling fact that roton-pair emission seems to be the dominant dissipation mechanism for electron bubbles accelerated above $\mathbf{v}_L$ in superfluid helium [3]. We conclude with a discussion of Unruh's proposal [4] that a fluid might offer a laboratory realization of black hole evaporation.

II. CONTINUOUS COLLAPSE DYNAMICS

Our system consists of N identical bosons interacting with each other and a static external potential (representing the walls). In the position representation the Hamiltonian is

$$\hat{H}=-\frac{1}{2M}\sum_{i=1}^{N}(\nabla_i^2)+V(\mathbf{r}_1,\ldots,\mathbf{r}_N)\ , \tag{1}$$

where V is symmetrical in all its arguments, M is the boson mass, and $\hbar=1$. The ground state wave function Ψ_0 is assumed known and is used to construct a class of wave functions having the same degrees of freedom as a *single* particle:

$$\Psi_\alpha=\exp\left[\int d^3\mathbf{r}\,\hat{\rho}(\mathbf{r})\alpha(\mathbf{r},t)\right]\Psi_0\ . \tag{2}$$

Here

$$\hat{\rho}(\mathbf{r})=\sum_{i=1}^{N}\delta^3(\mathbf{r}-\mathbf{r}_i) \tag{3}$$

is the density operator, and the function $\alpha(\mathbf{r},t)$ parametrizes our class of wave functions.

Originally introduced by Feynman [5], Ψ_α combines two important degrees of freedom of the superfluid. Considering for the moment a translationally invariant system, a Galilean transformation of the ground state is realized by the choice

$$\alpha(\mathbf{r},t)=iM\mathbf{v}_0\cdot\mathbf{r}\ , \tag{4}$$

where $\mathbf{v}_0$ is the flow velocity. Second, if $\alpha(\mathbf{r},t)$ has the form of a plane wave, then the wave function

$$\left[\int d^3\mathbf{r}\,\hat{\rho}(\mathbf{r})\alpha(\mathbf{r},t)\right]\Psi_0 \tag{5}$$

represents a density wave excitation while Ψ_α is the corresponding coherent state. As we shall see below, the flow and excitational degrees of freedom are no longer independent when the flow is nonuniform.

The form of the wave function Ψ_α is our main approximation. As already mentioned, this wave function is not new. What is different is how we proceed. Rather than minimize an energy expectation value to determine $\alpha(\mathbf{r},t)$, we use the "continuous collapse" scheme to define the dynamics of this function. A general discussion of this method, sometimes called the "time-dependent variational method," can be found in Ref. [2]; here we just sketch the idea. An initial state Ψ_α, defined by some $\alpha(\mathbf{r},0)$, is evolved forward an infinitesimal time $\Delta \mathbf{t}$ by the exact Schrödinger equation. The resulting state, in general, is not exactly of the same form and is "collapsed" into the state Ψ_α where $\alpha(\mathbf{r},\Delta t)$ is chosen so as to maximize the overlap. Up to an overall phase, this determines the time evolution of $\alpha(\mathbf{r},t)$. A detailed derivation for the case of the boson problem is given in Ref. [2]; here we simply state the result:

$$i\left[\sigma_\alpha \circ \frac{d\alpha}{dt}\right](\mathbf{r})=\frac{1}{2M}\{-\nabla\cdot[\rho_\alpha(\mathbf{r})\nabla\alpha(\mathbf{r})] +[\sigma_\alpha\circ|\nabla\alpha|^2](\mathbf{r})\} \ . \tag{6}$$

In addition to the explicit nonlinearities, the function $\alpha(\mathbf{r},t)$ also appears nonlinearly in the expectation values

$$\begin{aligned}\rho_\alpha(\mathbf{r})&=\langle\hat{\rho}(\mathbf{r})\rangle \ ,\\ \sigma_\alpha(\mathbf{r},\mathbf{r}')&=\langle\hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r}')\rangle-\langle\hat{\rho}(\mathbf{r})\rangle\langle\hat{\rho}(\mathbf{r}')\rangle \ ,\end{aligned} \tag{7}$$

where $\langle\ \rangle$ denotes an expectation value in the state Ψ_α. The symbol $\circ$ is the convolution operator:

$$(\sigma_\alpha\circ f)(\mathbf{r})=\int d^3\mathbf{r}'\sigma_\alpha(\mathbf{r},\mathbf{r}')f(\mathbf{r}') \ . \tag{8}$$

Equation (6) is the first in a series of approximate evolution equations for boson superfluids. More refined approximations follow from expanding the class of wave functions on which the dynamics is confined. A natural progression of wave functions [6] would include higher order density operator terms in the exponential of (2):

$$\int d^3\mathbf{r}\,\hat{\rho}(\mathbf{r})\alpha_1(\mathbf{r},t) +\tfrac{1}{2}\int d^3\mathbf{r}\int d^3\mathbf{r}'\hat{\rho}(\mathbf{r})\hat{\rho}(\mathbf{r}')\alpha_2(\mathbf{r},\mathbf{r}',t)+\cdots \ . \tag{9}$$

The consequences of keeping just the lowest order term are already quite interesting and unlikely to change qualitatively when the higher order terms are included.

Our time evolution equation differs significantly from another equation that is frequently used as a model of superfluids: the Gross-Pitaevskii or nonlinear Schrödinger equation [7]. The latter also describes the evolution of a complex-valued function, in this case the condensate wave function in the weak interaction limit. Unlike the Gross-Pitaevskii equation, however, the present equation contains nonlocal terms involving the pair correlation function σ_α. Although nonlocal modifications of the Gross-Pitaevskii equation have been considered as a way of correcting its microscopic properties [8], such as adding a roton minimum to the dispersion of excitations, the justification for such modifications is purely phenomenological. Another advantage of the present formulation is the possibility of systematic improvements. Such improvements unavoidably call for the introduction of new degrees of freedom, a program that can be in principle be carried out systematically using the density operator expansion above.

III. STATIC SOLUTIONS

We will be interested in static solutions of Eq. (6) that are characterized by irrotational flow fields $\mathbf{v}(\mathbf{r})$. To see how these time-independent solutions arise, we write

$$\alpha_{\mathbf{v}}(\mathbf{r})=\chi(\mathbf{r})+i\Phi(\mathbf{r}) \ , \tag{10}$$

where χ and Φ are real, and examine the imaginary and real parts of (6) separately:

$$0=-\nabla\cdot[\rho_\chi(\mathbf{r})\nabla\Phi(\mathbf{r})] \ , \tag{11a}$$

$$0=-\nabla\cdot[\rho_\chi(\mathbf{r})\nabla\chi(\mathbf{r})]+[\sigma_\chi\circ(|\nabla\chi|^2+|\nabla\Phi|^2)](\mathbf{r}) \ . \tag{11b}$$

The expectation values ρ and σ have been given the subscript χ to emphasize the fact that they depend only on the real part of $\alpha_{\mathbf{v}}$.

Solutions to (11) can be found using an iterative approach that begins with the inputs $\chi\approx 0$ and

$$\nabla\Phi_1(\mathbf{r})=M\mathbf{v}(\mathbf{r}) \ , \tag{12}$$

where the irrotational flow field $\mathbf{v}(\mathbf{r})$ satisfies

$$\nabla\cdot[\rho_0(\mathbf{r})\mathbf{v}(\mathbf{r})]=0 \ . \tag{13}$$

This solves (11a). We next refine χ by considering (11b). Treating $\mathbf{v}$ as small and keeping only the leading order terms,

$$0=-\nabla\cdot[\rho_0(\mathbf{r})\nabla\chi_2(\mathbf{r})]+M^2[\sigma_0\circ|\mathbf{v}|^2](\mathbf{r}) \ , \tag{14}$$

we see that χ_2 is the solution of a generalized Poisson's equation with a source term of order $|\mathbf{v}|^2$. Thus the corrections to ρ_χ (and σ_χ) are of the same order:

$$\begin{aligned}\rho_\chi(\mathbf{r})&=\rho_0(\mathbf{r})+\rho_2(\mathbf{r})+\cdots \ ,\\ \sigma_\chi(\mathbf{r},\mathbf{r}')&=\sigma_0(\mathbf{r},\mathbf{r}')+\sigma_2(\mathbf{r},\mathbf{r}')+\cdots \ .\end{aligned} \tag{15}$$

Returning now to (11a), the next-to-leading order terms produce another Poisson equation,

$$0=-M\nabla\cdot[\rho_2(\mathbf{r})\mathbf{v}(\mathbf{r})]-\nabla\cdot[\rho_0(\mathbf{r})\nabla\Phi_3(\mathbf{r})] \ , \tag{16}$$

giving a correction to Φ of order $|\mathbf{v}|^3$. Repeating this process we obtain an expansion of the solution in powers of $|\mathbf{v}|^2$:

$$\begin{aligned}\Phi&=\Phi_1+\Phi_3+\cdots \ ,\\ \chi&=\chi_2+\chi_4+\cdots \ .\end{aligned} \tag{17}$$

In a system with Galilean invariance, the special case $\mathbf{v}(\mathbf{r})=\text{const}$, $\chi=0$ corresponds to an exact solution. In particular, the source term in (14)—and consequently χ_2—vanishes because of the identity

$$\int d^3\mathbf{r}'\sigma_0(\mathbf{r},\mathbf{r}')=0 \ . \tag{18}$$

The true expansion parameter is therefore not $|\mathbf{v}|^2$, but rather $|\delta\mathbf{v}|^2$, where $\delta\mathbf{v}$ represents the deviation from uniform flow. This distinction is a significant one, since, as shown below, these static flow solutions are unstable at a finite critical mean velocity, but arbitrarily small deviations from uniformity.

IV. LINEAR STABILITY ANALYSIS

Viewed as solutions to classical nonlinear equations, the static flows found in the preceding section are viable only if they are stable to perturbations. The standard analysis proceeds by substituting

$$\alpha(\mathbf{r},t)=\alpha_{\mathrm{v}}(\mathbf{r})+\beta(\mathbf{r},t) \tag{19}$$

into Eq. (6) and retaining only terms linear in the perturbation β. For simplicity, we will in addition only consider terms in $\boldsymbol{\alpha}_{\mathrm{v}}$ to order $|\delta\mathbf{v}|$, so that corrections to the density and pair correlation function (of order $|\delta\mathbf{v}|^2$) can be neglected. On the other hand, corrections to the density that are linear in the perturbation must be included:

$$\rho_\alpha=\rho_0+\sigma_0\circ(\beta+\beta^*)+\cdots \ . \tag{20}$$

The corresponding correction to σ_α either leads to terms quadratic in β or terms of order $|\delta\mathbf{v}|^2$ and can be neglected. With these considerations in mind, one obtains the following equation for β:

$$i\sigma_0\circ\frac{d\beta}{dt}=H[\beta]+G[\beta^*]+O(|\delta\mathbf{v}|^2) \ , \tag{21}$$

$$H[\beta]=-\frac{1}{2M}\nabla\cdot[\rho_0\nabla\beta] -\frac{i}{2}\{\sigma_0\circ(\mathbf{v}\cdot\nabla\beta)+\nabla\cdot[\mathbf{v}(\sigma_0\circ\beta)]\} \ , \tag{22}$$

$$G[\beta^*]=\frac{i}{2}\{\sigma_0\circ(\mathbf{v}\cdot\nabla\beta^*)-\nabla\cdot[\mathbf{v}(\sigma_0\circ\beta^*)]\} \ . \tag{23}$$

Equation (21) is one of our main results. To understand its significance, we begin with a translationally invariant system where

$$\rho_0(\mathbf{r})=\rho_0 \ , \quad \sigma_0(\mathbf{r},\mathbf{r}')=\sigma_0(\mathbf{r}-\mathbf{r}') \ . \tag{24}$$

If we further confine our attention to a state of uniform flow, say $\mathbf{v}(\mathbf{r})=\mathbf{v}_0$, then

$$\nabla\cdot[\mathbf{v}_0(\sigma_0\circ\beta^*)]=\mathbf{v}_0\cdot(\sigma_0\circ\nabla\beta^*)=\sigma_0\circ(\mathbf{v}_0\cdot\nabla\beta^*) \ , \tag{25}$$

and the operator G exactly vanishes. Moreover, since the equation now has translational invariance, the normal mode solutions are plane waves:

$$\beta(\mathbf{r},t)=\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)] \ . \tag{26}$$

Substituting (26) into (21) we obtain a formula for the mode frequency,

$$\omega_{\mathbf{k}}=\epsilon_{\mathbf{k}}+\mathbf{k}\cdot\mathbf{v}_0 \ , \tag{27}$$

which is just the Galilean transformation of the Bijl-Feynman [5] quasiparticle energy:

$$\epsilon_{\mathbf{k}}=\frac{|\mathbf{k}|^2}{2MS(\mathbf{k})} \ . \tag{28}$$

Here $S(\mathbf{k})$ is the Fourier transform of the pair correlation function,

$$S(\mathbf{k})=\frac{1}{\rho_0}\int d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\sigma_0(\mathbf{r}) \ , \tag{29}$$

and the source of structure, such as a dip, in the dispersion relation. Given the form of our many-body wave function, this result is not really surprising. Its deficiencies—such as the overestimate, in the case of helium, of the energy of the "roton dip"—can be corrected by refining the wave function as indicated in (9) [6].

We next consider the effects of nonuniformity in the flow. Because of the symmetrized form of the velocity dependent terms, the operator H continues to be Hermitian when the flow is nonuniform. Normal modes $\gamma_i(\mathbf{r})$ are defined by the eigenvalue equation,

$$H[\gamma_i]=\omega_i(\sigma_0\circ\gamma_i) \ . \tag{30}$$

It is easily verified that

$$(\gamma_i,H[\gamma_j])=(H[\gamma_i],\gamma_j) \ , \tag{31}$$

where (,) denotes the usual Hermitian inner product. Consequently, the eigenvalues ω_i are real and orthogonality takes the form

$$(\gamma_i,\sigma_0\circ\gamma_j)=\int d^3\mathbf{r}\int d^3\mathbf{r}'\gamma_i^*(\mathbf{r})\sigma_0(\mathbf{r},\mathbf{r}')\gamma_j(r') \ , \quad =\delta_{ij} \ . \tag{32}$$

Nonuniformity in the flow also forces us to consider the operator G. Being of order $|\delta\mathbf{v}|$, it might seem appropriate to treat G as a perturbation. For modes with small frequency, however, this would be dangerous since G couples β with β^*, i.e., a mode with its negative frequency counterpart. In other words, the problem of degenerate perturbation theory can arise for a single mode coupled to its time-reversed self. We will deal with this problem by considering a situation in which only a single mode, $i=1$, has a small frequency and is well separated from all the other modes. As discussed below, modes localized near protrusions are expected to have this property [9].

An arbitrary perturbation can always be expanded in terms of the complete set of orthonormal modes of H:

$$\beta(\mathbf{r},t)=\sum_i c_i(t)\gamma_i(\mathbf{r}) \ . \tag{33}$$

Substituting this expansion into (21) and taking inner products with the various modes yields a matrix equation for the coefficient functions $c_i(t)$. Considering only those basis states for which the effect of the operator G might be large, we confine our attention to the simple 1×1 matrix equation that includes only the single mode $i=1$. This equation reads

$$i\frac{dc_1}{dt}=\omega_1 c_1+\Gamma_{11}c_1^* , \tag{34}$$

where

$$\Gamma_{11}=(\gamma_1,G[\gamma_1^*])$$
$$=\frac{i}{2}\int d^3\mathbf{r}\int d^3\mathbf{r}'\{\gamma_1^*(\mathbf{r})\sigma_0(\mathbf{r},\mathbf{r}')[\mathbf{v}(\mathbf{r}')-\mathbf{v}(\mathbf{r})]\cdot\nabla\gamma_1^*(\mathbf{r}')-\gamma_1^*(\mathbf{r})\sigma_0(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{v}(\mathbf{r}')\gamma_1^*(\mathbf{r}')\} . \tag{35}$$

This shows explicitly that the coupling between a mode and its time reversal is of order $|\delta\mathbf{v}|$ in the flow nonuniformity. The general solution to (34) is given by

$$c_1(t)=\lambda\sqrt{-\Gamma_{11}}\left[\sqrt{\omega_1+\omega}\,e^{-i\omega(t-\tau)}+\sqrt{\omega_1-\omega}\,e^{i\omega(t-\tau)}\right] , \tag{36}$$

where

$$\omega^2=\omega_1^2-|\Gamma_{11}|^2 , \tag{37}$$

and the two real parameters λ and τ are arbitrary. This is the second of our main results. It shows that there is a window of instability determined by the condition $|\omega_1|<|\Gamma_{11}|$.

V. DISCUSSION

We now consider the situation frequently encountered in real experiments with superfluid helium, where the flow field varies adiabatically. A particularly clean example is an electron bubble being accelerated by a weak electric field [3]. Treating the bubble as a large classical object, the flow from the point of view of a comoving observer is quasistatic, although adiabatically growing in magnitude with time. Because the flow velocity near the equator of the bubble exceeds the velocity far from the bubble by as much as 50%, we expect localized, rotonlike states to be found there [9]. Localization can be argued in the semiclassical limit where Eq. (27) applies and has also been verified by Lenosky and Elser [10] in explicit solutions to the mode equation (30). In the present discussion, what most interests us is the adiabatic variation of the frequency ω_1 of this localized mode as the asymptotic flow velocity is increased. When the Landau criterion is satisfied in the region of fast flow, we expect ω_1 to pass through zero into negative values. This is the situation analyzed above, where the frequency ω of the perturbation is given by (37). A graph of ω vs ω_1 (see Fig. 1) resembles the crossing of two quantum mechanical energy levels, but also differs from this in an essential way. Identifying ω with the decreasing energy of a one-roton state and the abscissa ($\omega=0$) as the zero-roton or "vacuum" energy level, the crossing ($\omega=\omega_1$) marks where the system could make a transition from one to the other. Unlike the level crossing problem, however, here the self-coupling term Γ_{11} leads to an *attraction* of levels, rather than repulsion. This leaves a finite range of ω_1 where the notion of a one-roton state breaks down completely, since its energy would be imaginary. A more direct interpretation is that the flow state itself is unstable, so that an excitation spectrum has no meaning.

The apparent attraction of energy levels can be understood if the notion of "excitation" is made more precise. First consider the complete set of coupled mode equations:

$$i\frac{dc_i}{dt}=\omega_i c_i+\sum_j\Gamma_{ij}c_j^* . \tag{38}$$

With each mode we associate an operator $\hat{a}_i$ whose expectation value is the mode amplitude:

$$\langle\hat{a}_i\rangle=c_i(t) ,$$
$$\langle\hat{a}_i^\dagger\rangle=c_i^*(t) . \tag{39}$$

Here expectation values are computed in the Fock space generated by the complete set of creation operators $\hat{a}_i^\dagger$ and the usual commutation relations hold:

$$[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij} . \tag{40}$$

With these definitions, Eq. (38) can be viewed as the Heisenberg equations of motion,

$$i\frac{d}{dt}\langle\hat{a}_i\rangle=\langle[\hat{a}_i,\hat{H}]\rangle , \tag{41}$$

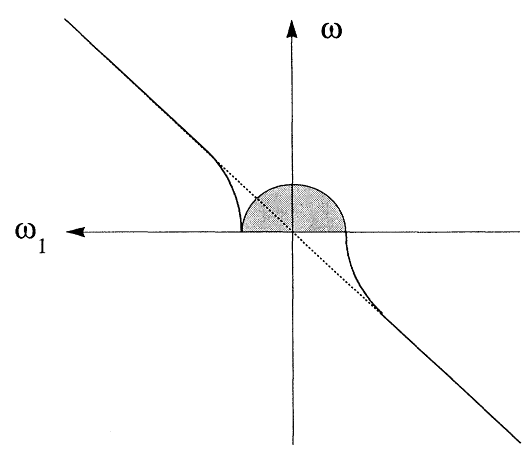

FIG. 1. Roton-vacuum "level crossing" shows attraction, rather than repulsion of levels. Over a small interval the vacuum is unstable; the growth rate of the unstable mode is shown shaded.

for the Hamiltonian

$$\hat{H}=\sum_i \omega_i \hat{a}_i^\dagger \hat{a}_i+\tfrac{1}{2}\sum_{ij}\left[\Gamma_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger+\Gamma_{ij}^*\hat{a}_i\hat{a}_j\right] . \tag{42}$$

This shows immediately that the vacuum and one-roton states are not even coupled by the term Γ, but rather, that this term has the interpretation of changing the roton number by *two*. In the single mode ($i=1$) problem considered above, the states with occupation numbers $n=0$ and $n=1$ still have "attracting" energy levels. It is clear now, however, that this attraction is really an artifact of the repulsion of $n=1$ from $n=3$ being stronger than the repulsion of $n=0$ from $n=2$. Finally, the classical instability for $|\omega_1|<|\Gamma_{11}|$ is consistent with the fact that the Fock space Hamiltonian is no longer bounded from below in this parameter range.

The roton-pair creation interpretation of the superflow instability is especially interesting in light of electron-bubble drift velocity measurements by McClintock and Bowley [3]. These measurements show rather conclusively that the dissipation process is best modeled as the coherent emission of roton pairs when the bubble exceeds the Landau velocity $\mathbf{v}_L$. Moreover, from the simple fact that the bubbles can easily be accelerated to velocities significantly exceeding $\mathbf{v}_L$, one infers that the coupling is weak and can be treated perturbatively. The relevant operator is written as [3]

$$\frac{1}{\Omega}\sum_{\mathbf{k}_1,\mathbf{k}_2} V_{\mathbf{k}_1,\mathbf{k}_2}\hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2}^\dagger e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\hat{\mathbf{R}}} , \tag{43}$$

where $\hat{\mathbf{R}}$ is the bubble's position operator and Ω is the system volume. Near threshold, where both rotons have wave vectors close to the critical wave vector $\mathbf{k}_0$, the experiment measures the pair-creation matrix element $V_{\mathbf{k}_0,\mathbf{k}_0}$.

In the limit that the electron bubble is very massive, recoil effects are small and the pair-creation matrix element reduces to the coupling term Γ in our analysis of time-independent flow. It is now appropriate to introduce continuum modes $\gamma_{\mathbf{k}}$ which approach plane waves far from the bubble. These modes satisfy

$$H[\gamma_{\mathbf{k}}]=\omega_{\mathbf{k}}(\sigma_0\circ\gamma_{\mathbf{k}}) \tag{44}$$

with $\omega_{\mathbf{k}}$ given by (27). The proper normalization ,

$$(\gamma_{\mathbf{k}},\sigma_0\circ\gamma_{\mathbf{k}'})=\frac{(2\pi)^3}{\Omega}\delta^3(\mathbf{k}-\mathbf{k}') , \tag{45}$$

is conveniently expressed using modes $\bar{\gamma}_{\mathbf{k}}$:

$$\gamma_{\mathbf{k}}(\mathbf{r})=\frac{\bar{\gamma}_{\mathbf{k}}(\mathbf{r})}{\sqrt{\Omega\rho_0 S(\mathbf{k})}} , \tag{46}$$

where

$$\bar{\gamma}_{\mathbf{k}}(\mathbf{r})\sim 1\times\exp[i(\mathbf{k}\cdot\mathbf{r})], \quad |\mathbf{r}|\to\infty . \tag{47}$$

With these definitions, the pair-creation matrix element is given by

$$V_{\mathbf{k}_1,\mathbf{k}_2}=\frac{i}{4\rho_0\sqrt{S(\mathbf{k}_1)S(\mathbf{k}_2)}}\int d^3\mathbf{r}\int d^3\mathbf{r}'\{\bar{\gamma}_{\mathbf{k}_1}^*(\mathbf{r})\sigma_0(\mathbf{r},\mathbf{r}')[\mathbf{v}(\mathbf{r}')-\mathbf{v}(\mathbf{r})]\cdot\nabla\bar{\gamma}_{\mathbf{k}_2}^*(\mathbf{r}')-\bar{\gamma}_{\mathbf{k}_1}^*(\mathbf{r})\sigma_0(\mathbf{r},\mathbf{r}')\nabla\cdot\mathbf{v}(\mathbf{r}')\bar{\gamma}_{\mathbf{k}_2}^*(\mathbf{r}')\} . \tag{48}$$

The evaluation of this matrix element in the bubble geometry is currently underway [10]. Because our wave function (2) predicts a roton energy (28) that is off by about a factor of 2, we expect at best a similar level of agreement with the experimentally measured matrix element. Nevertheless, it is satisfying that the pair-creation mechanism now appears to be a very general feature of an unstable superfluid.

The problem of the decay of superflow by the emission of excitations bears some similarities with the evaporation of black holes by the Hawking process [11]. In both situations we deal with a static field configuration (respectively, flow and gravity) which represent a very high degree of excitation above the "vacuum" ($\mathbf{v}=0$ and flat space, respectively). Some notion of "general covariance" is evident in a fluid as well. Namely, to the extent that each atom is subject only to forces from atoms in its immediate neighborhood, the physics inside a small comoving volume should be indistinguishable from the physics in a volume of fluid at rest. The analogy has been taken the furthest by Unruh [4], who explicitly constructed a mapping between the classical, hydrodynamical equations of convergent fluid flow, and the Schwarzschild geometry of a black hole. When the reasoning leading to Hawking's thermal spectrum is applied to the fluid problem, not surprisingly, a thermal spectrum of radiated sound is predicted.

Unruh's treatment differs from the present one in at least three ways. First, his equations apply only at long length scales and hence do not take into consideration the fact that in a medium such as helium, microscopic excitations (rotons) are the first to go unstable. In other words, a "rotonic" horizon would be formed long before his "sonic" horizon. This corresponds, in the black hole problem, to an evaporation process dominated by Planck scale physics. On the other hand, one could imagine a superfluid without a "roton dip," where Unruh's long wavelength description would be adequate.

The second basic difference in the two approaches is in the application of quantum mechanics. Unruh begins with classical equations and applies quantization only at the end when he considers the linearized equations for the perturbations about the background flow. In the present approach, quantum mechanics is introduced at the very beginning, so that even the background flow is described by a wave function. The difference here is not merely technical in nature, but in fact points out the main challenge posed by the black hole problem, namely, how, in fact, *does* one quantize in a curved space-time? In the fluid problem one has the luxury of having an ambient, flat space-time, where quantum mechanics is clearly defined. This should be contrasted with the situation of the black hole physicist, who is forced to make bold proclamations, in particular, defining "vacuum" as the particle state in the neighborhood of an observer freely falling through the event horizon [4,11].

Finally, the experimental realizations that have been emphasized in the two approaches also differ in significant ways. Unruh has considered spherically symmetric flow which requires some sort of tube to carry away the converging fluid. The flow inside this tube would be supersonic and highly unstable. Classically, the (possibly messy) details of this flow are irrelevant in that the fluid inside the tube is causally disconnected from the flow outside. In the Hawking process, however, one considers modes that extend into both regions and are therefore sensitive to such details. Perhaps the detail that is most troublesome is the balance of energy: as energy in the form of excitations is being emitted, the energy in the background flow must somehow decrease. The decay of flow around a moving bubble is much more controlled in this respect. Here the energy in the background flow is simply the kinetic energy of the bubble,

$$E_B(\mathbf{P})=\frac{|\mathbf{P}|^2}{2M_B} , \tag{49}$$

where M_B is the (mostly hydrodynamic) bubble mass. The energy of the emitted roton pair is then accounted for by the decrease in $E_B(\mathbf{P})$ due to recoil:

$$\begin{aligned}\Delta E_B &= E_B(\mathbf{P})-E_B(\mathbf{P}-\mathbf{k}_1-\mathbf{k}_2) ,\\ &\cong \mathbf{v}_B\cdot(\mathbf{k}_1+\mathbf{k}_2)=\epsilon_{\mathbf{k}_1}+\epsilon_{\mathbf{k}_2} .\end{aligned} \tag{50}$$

Here the last line has a solution for the two roton momenta provided the bubble velocity $\mathbf{v}_B=\mathbf{P}/M_B$ exceeds the Landau velocity.

In view of these differences, it is fair to say that a considerable gulf separates the understanding of superflow instability developed in this paper, and Unruh's treatment [4] inspired by the Hawking process [11]. Until this gulf is successfully crossed, it is premature to claim that Hawking radiation has been observed in the laboratory.

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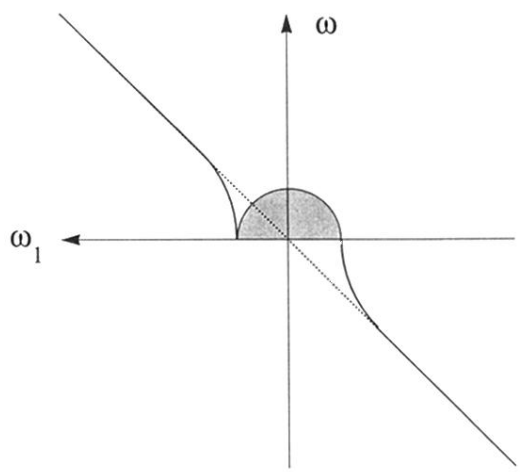

FIG. 1. Roton-vacuum "level crossing" shows attraction, rather than repulsion of levels. Over a small interval the vacuum is unstable; the growth rate of the unstable mode is shown shaded.